Holographic Transformation, Belief Propagation and Loop Calculus for Generalized Probabilistic Theories

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Holographic transformation, belief propagation and loop calculus

Holographic transformation [Valiant 2004]: Linear-algebraic techinique for deriving non-trivial equalities between sum of products.

$$\sum_{\mathbf{x}\in\{0,1\}^n} \prod_a f_a(\mathbf{x}_{(a)}) = \sum_{\mathbf{y}\in\{0,1\}^m} \prod_b g_b(\mathbf{y}_{(b)})$$

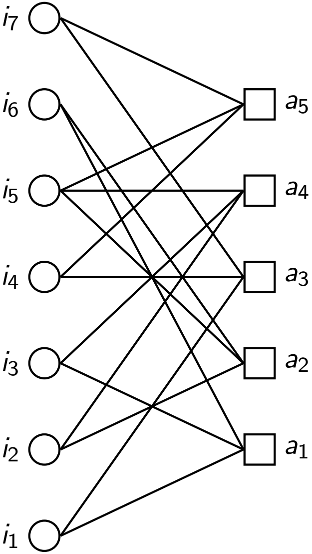
MacWilliams identity, high temperature expansion, loop calculus ...

Belief propagation [Pearl 1982]: Efficient message-passing algorithm for approximating marginal distribution and partition function.

Loop calculus [Chartkov and Chernyak 2006]: Equality relating partiton function and the approximation obtained by BP.

$$\sum_{\mathbf{x}} \prod_a f_a(\mathbf{x}_{(a)}) = Z_{\mathrm{BP}} \left(1 + \sum_{\text{loop structure } L} \mathcal{K}(L)\right).$$

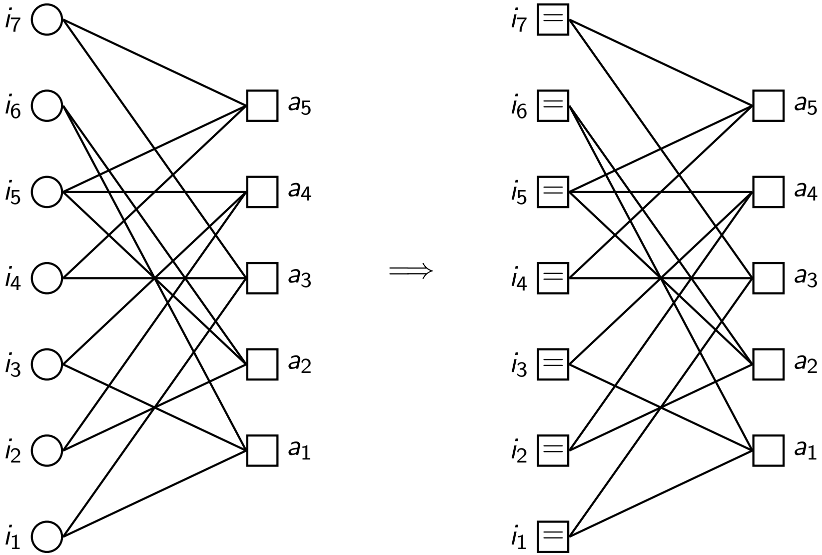
Factor graph and partition function



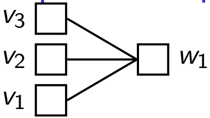
$$Z(G) := \sum_{(x_1,\dots,x_7)\in\mathcal{X}^7} f_1(x_1, x_3, x_6) \cdot f_2(x_2, x_5, x_6) f_3(x_1, x_4, x_7) \cdot f_4(x_2, x_3, x_5) f_5(x_4, x_5, x_7)$$

partition function

Bipartite normal factor graph



Inner product representation



$$\sum_{(x_1,x_2,x_3)\in\{0,1\}^3} f_1(x_1)f_2(x_2)f_3(x_3)g(x_1,x_2,x_3)$$

$$= \text{sum of all elements of } \begin{bmatrix} f_1(0)f_2(0)f_3(0)g(0,0,0) \\ f_1(0)f_2(0)f_3(1)g(0,0,1) \\ \vdots \\ f_1(1)f_2(1)f_3(1)g(1,1,1) \end{bmatrix}$$

$$= \text{inner product of } \begin{bmatrix} f_1(0)f_2(0)f_3(0) \\ \vdots \\ f_1(1)f_2(1)f_3(1) \end{bmatrix} \text{ and } \begin{bmatrix} g(0,0,0) \\ \vdots \\ g(1,1,1) \end{bmatrix}$$

$$= \text{inner product of } \begin{bmatrix} f_1(0) \\ f_1(1) \end{bmatrix} \otimes \begin{bmatrix} f_2(0) \\ f_2(1) \end{bmatrix} \otimes \begin{bmatrix} f_3(0) \\ f_3(1) \end{bmatrix} \text{ and } \begin{bmatrix} g(0,0,0) \\ \vdots \\ g(1,1,1) \end{bmatrix}$$

Inner product representation

$$F_v := \sum_{\mathbf{x}_{\partial v} \in \prod_{w \in \partial v} \mathcal{X}_{v,w}} f_v(\mathbf{x}_{\partial v}) \bigotimes_{w \in \partial v} e^{v,w}_{x_{v,w}}.$$

The vector $G_w \in \mathcal{V}_{\partial w}$ is also defined in the same way. It holds

$$\bigotimes_{v \in V} F_v = \sum_{\mathbf{x} \in \mathcal{X}} \prod_{v \in V} f_v(\mathbf{x}_{\partial v}) \bigotimes_{(v,w) \in E} e^{v,w}_{x_{v,w}}$$

$$\bigotimes_{w \in W} G_w = \sum_{\mathbf{x} \in \mathcal{X}} \prod_{w \in W} g_w(\mathbf{x}_{\partial w}) \bigotimes_{(v,w) \in E} e^{v,w}_{x_{v,w}}.$$

$$\left\langle \bigotimes_{v \in V} F_v, \bigotimes_{w \in W} G_w \right\rangle = \sum_{\mathbf{x}} \prod_{v \in V} f_v(\mathbf{x}_{\partial v}) \prod_{w \in W} g_w(\mathbf{x}_{\partial w})$$

A partition function is an inner product.

Adjoint map

A: Linear map $\mathcal{V} \to \mathcal{V}'$.

A^*: Adjoint map $\mathcal{V}' \to \mathcal{V}$ of linear map $A \overset{\text{def}}{\iff}$

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle, \qquad \forall x \in \mathcal{V},\ y \in \mathcal{V}'$$

Adjoint map $\iff$ transpose of the matrix

Holographic transformation

Theorem (Holant theorem for the bipartite model)

Let $\Phi_{v,w}$ be an invertible linear map on $\mathcal{V}_{v,w}$ and $\hat{\Phi}_{v,w}$ be the inverse map of $\Phi_{v,w}$ for $(v,w) \in E$. Then, it holds

$$\left\langle \bigotimes_{v\in V} f_v, \bigotimes_{w\in W} g_w \right\rangle_{\mathcal{V}} = \left\langle \bigotimes_{v\in V} \hat{f}_v, \bigotimes_{w\in W} \hat{g}_w \right\rangle_{\mathcal{V}}$$

where

$$\hat{f}_v = \left(\bigotimes_{w\in\partial v} \hat{\Phi}_{v,w} \right)(f_v), \qquad \hat{g}_w = \left(\bigotimes_{v\in\partial w} \Phi^*_{v,w} \right)(g_w).$$

Proof.

$$\left\langle \bigotimes_{v\in V} f_v, \bigotimes_{w\in W} g_w \right\rangle_{\mathcal{V}} = \left\langle \left(\bigotimes_{(v,w)\in E} \Phi_{v,w} \circ \hat{\Phi}_{v,w} \right) \left(\bigotimes_{v\in V} f_v \right), \bigotimes_{w\in W} g_w \right\rangle_{\mathcal{V}}$$

$$= \left\langle \left(\bigotimes_{(v,w)\in E} \hat{\Phi}_{v,w} \right) \left(\bigotimes_{v\in V} f_v \right), \left(\bigotimes_{(v,w)\in E} \Phi^*_{v,w} \right) \left(\bigotimes_{w\in W} g_w \right) \right\rangle_{\mathcal{V}} = \left\langle \bigotimes_{v\in V} \hat{f}_v, \bigotimes_{w\in W} \hat{g}_w \right\rangle_{\mathcal{V}}.$$

□

Quantum bipartite model

When in the inner product model,

- The linear spaces: The set of $k \times k$ Hermitian matrices.
- The inner product: The Hilbert-Schmidt inner product, i.e., $\langle A, B \rangle := \mathrm{Tr}(AB)$.

$$\left\langle \bigotimes_{v \in V} \omega_v, \bigotimes_{w \in W} P_w \right\rangle_{\mathcal{V}} = \mathrm{Tr}\left(\bigotimes_{v \in V} \omega_v \bigotimes_{w \in W} P_w \right).$$

Probability of getting $\bigotimes_{w \in W} P_w$ on the state $\bigotimes_{v \in V} \omega_v$.

Equivalence between Schrödinger picture and Heisenberg picture is special case of the Holographic transformation

$$\left\langle \bigotimes_{v \in V} \left(\bigotimes_{w \in \partial v} T_{v,w} \right) (\omega_v), \bigotimes_{w \in W} P_w \right\rangle_{\mathcal{V}}$$
$$= \left\langle \bigotimes_{v \in V} \omega_v, \bigotimes_{w \in W} \left(\bigotimes_{v \in \partial w} T^*_{v,w} \right) (P_w) \right\rangle_{\mathcal{V}}$$

Generalized probabilistic theories

C : convex cone.

$u \in$ interior of $C^* := \{x \in V \mid \langle x, y\rangle \geq 0, \forall y \in C\}$.

$$\text{Set of states} = \{\omega \in V \mid \omega \in C, \langle \omega, u\rangle = 1\}.$$

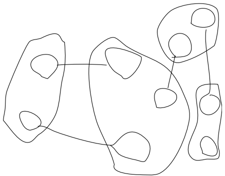
$$\text{Set of effects} = \{e \in V \mid e \in C^*, u - e \in C^*\}.$$

$$\text{Set of measurements} = \{(e_1, \ldots, e_k) \mid e_1 + \cdots + e_k = u,\ k = 1, 2, 3, \ldots\}$$

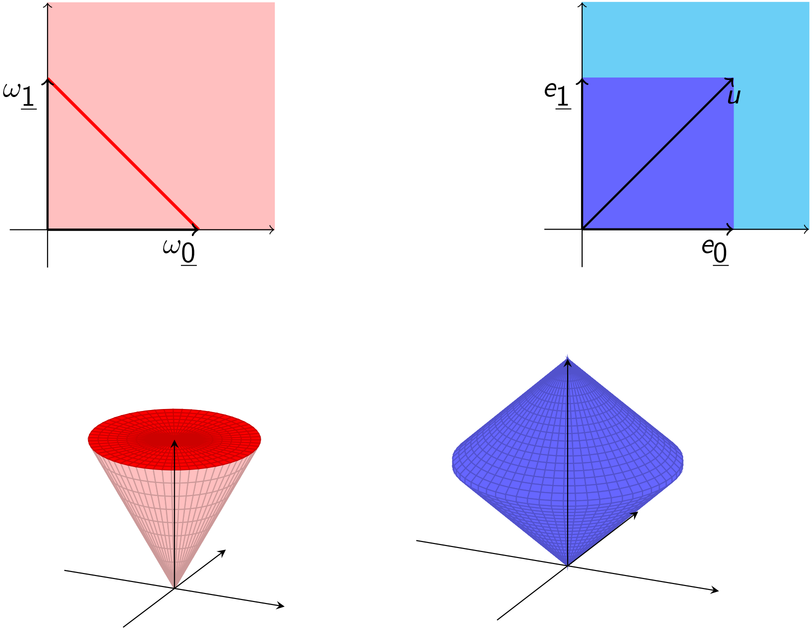
Probability of outcome i is equal to $\langle \omega, e_i\rangle$.

- Classical theory: C is the set of non-negative vectors.
- Quantum theory: C is the set of PSD matrices.

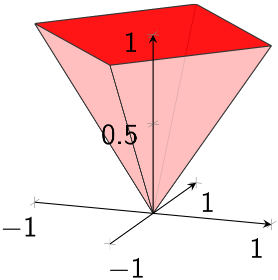
$$\left\langle \bigotimes_{v \in V} \omega_v, \bigotimes_{w \in W} e_w \right\rangle_{\mathcal{V}}.$$

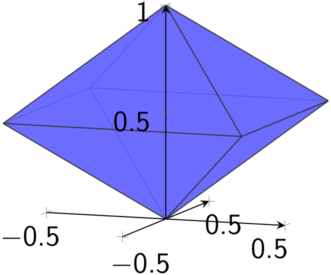


Classical and quantum theory



Toy model: gbit





$$\omega_0 = \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \qquad \omega_1 = \begin{bmatrix}0 & 1 & 1\end{bmatrix},$$
$$\omega_2 = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}, \qquad \omega_3 = \begin{bmatrix}0 & -1 & 1\end{bmatrix}.$$

$$e_0 = \frac{1}{2}\begin{bmatrix}1 & 1 & 1\end{bmatrix}, \qquad e_1 = \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\end{bmatrix},$$
$$e_2 = \frac{1}{2}\begin{bmatrix}1 & -1 & 1\end{bmatrix}, \qquad e_3 = \frac{1}{2}\begin{bmatrix}-1 & -1 & 1\end{bmatrix},$$
$$u = \begin{bmatrix}0 & 0 & 1\end{bmatrix}.$$

Beleif propagation for GPT

Definition (Belief propagation for GPT)

Let $(m^{(0)}_{v\to w} \in C_{v,w})_{(v,w)\in E}$ be arbitrarily chosen initial messages. Then, in the belief propagation, the messages are updated according to the following rules

$$m^{(t)}_{v\to w} = \frac{1}{Z^{(t)}_{v\to w}} \left\langle \omega_v, \bigotimes_{w'\in\partial v\setminus\{w\}} m^{(t)}_{w'\to v} \right\rangle_{\mathcal{V}_{\partial v\setminus w}}$$

$$m^{(t)}_{w\to v} = \frac{1}{Z^{(t)}_{w\to v}} \left\langle \bigotimes_{v'\in\partial w\setminus\{v\}} m^{(t-1)}_{v'\to w}, e_w \right\rangle_{\mathcal{V}_{\partial w\setminus v}}$$

for $t = 1, 2, \ldots$ for all $(v, w) \in E$ where the strictly positive constants $Z^{(t)}_{v\to w}$ and $Z^{(t)}_{w\to v}$ are chosen such that $\langle m^{(t)}_{v\to w}, u^*_{v,w}\rangle_{\mathcal{V}_{v,w}} = 1$ and $\langle u_{v,w}, m^{(t)}_{w\to v}\rangle_{\mathcal{V}_{v,w}} = 1$, respectively.

Loop calculus for GPT

Theorem (Loop calculus for GPT)

For any fixed point $(m_{v\to w}, m_{w\to v})_{(v,w)\in E}$ *of BP,*

$$\left\langle \bigotimes_{v\in V} \omega_v, \bigotimes_{w\in W} e_w \right\rangle_{\mathcal{V}} = Z_{\mathrm{BP}} \left(1 + \sum_{L\text{: generalized loop}} \mathcal{K}(L) \right)$$

where Z_{BP} *denotes the Bethe appoximation for GPT, i.e.,*

$$Z_{\mathrm{BP}} := \prod_{v\in V} \left\langle \omega_v, \bigotimes_{w\in\partial v} m_{w\to v} \right\rangle_{\mathcal{V}_{\partial v}} \prod_{w\in W} \left\langle \bigotimes_{v\in\partial w} m_{v\to w}, e_w \right\rangle_{\mathcal{V}_{\partial w}} \cdot \prod_{(v,w)\in E} \frac{1}{\langle m_{v\to w}, m_{w\to v} \rangle_{\mathcal{V}_{v,w}}}.$$

Summary

- A partition function is understood as an inner product.
- Holographic transformation (Holant theorem) can be understood by the inner product representation and adjoint map.

- Probability of locally factorized effect on locally factorized state of GPT is a natural instance of the inner product model.
- Belief propagation, the Bethe approximation and loop calculus for GPT are straightforwardly obtained.

Everything is linear-algebraic !

Future work: Application, e.g., syndrome decoding of stabilizer codes and simulation of MBQC.

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