Growth Rate of Spatially Coupled LDPC codes

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Workshop on Spatially Coupled Codes and Related Topics

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Contents

1. Factor graph, Bethe approximation and belief propagation
2. Relation between annealed free energy and belief propagation
3. Growth rate of spatially coupled LDPC codes and threshold saturation phenomenon

Here, growth rate is

$$G(\omega) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\omega)]$$

$Z(\omega)$: the number of codewords of relative weight $\omega \in [0,1]$.

Factor graph, Bethe approximation and belief propagation

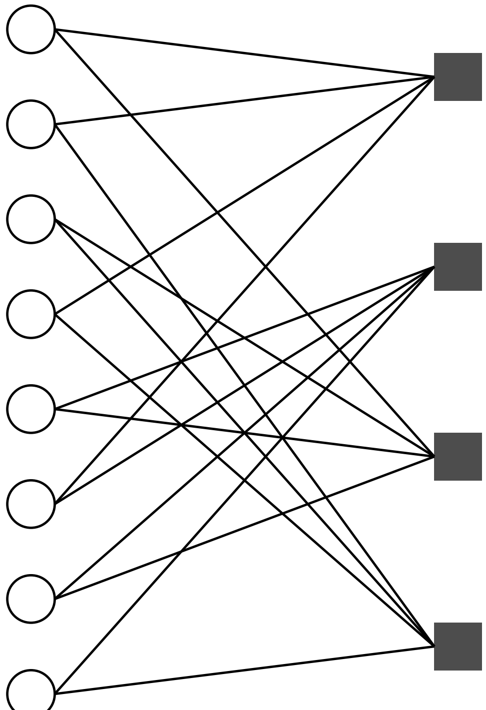
Factor graph

Factor graph: bipartite graph which defines probability measure

$$p(\boldsymbol{x}) = \frac{1}{Z} \prod_a f_a(\boldsymbol{x}_{\partial a})$$

$$Z := \sum_{\boldsymbol{x} \in \mathcal{X}^n} \prod_a f_a(\boldsymbol{x}_{\partial a}), \qquad \text{(partition function)}$$

$f_a(\boldsymbol{x}_{\partial a}) : \mathcal{X}^{r_a} \to \mathbb{R}_{\geq 0}$



Gibbs free energy

$$p(\boldsymbol{x}) = \frac{1}{Z} \prod_a f_a(\boldsymbol{x}_{\partial a})$$

Approximation by simple distribution q of p which is defined by factor graph

$$\begin{aligned}
D(q \| p) &= \sum_{\boldsymbol{x}} q(\boldsymbol{x}) \log \frac{q(\boldsymbol{x})}{p(\boldsymbol{x})} \\
&= \log Z - \sum_{\boldsymbol{x}} q(\boldsymbol{x}) \log \left(\prod_a f_a(\boldsymbol{x}_{\partial a}) \right) + \sum_{\boldsymbol{x}} q(\boldsymbol{x}) \log q(\boldsymbol{x}) \\
&=: \log Z + U(q) - H(q) \\
&=: \log Z + F_{\mathrm{Gibbs}}(q)
\end{aligned}$$

$U(p)$: internal energy
$H(p)$: entropy
$F_{\mathrm{Gibbs}}(p)$: Gibbs free energy

Mean field approximation and Bethe approximation

Mean field approximation

$$q(\boldsymbol{x}) = \prod_i b_i(x_i)$$

Degree of freedom is reduced from q^n to nq

Bethe approximation

$$q(\boldsymbol{x}) = \frac{\prod_a b_a(\boldsymbol{x}_{\partial a})}{\prod_i b_i(x_i)^{d_i - 1}}$$

d_i: degree of variable node i

When factor graph is tree, Bethe approximation can be exact

Bethe free energy

$$U(q) = -\sum_{\boldsymbol{x}} q(\boldsymbol{x}) \log \left(\prod_a f_a(\boldsymbol{x}_{\partial a}) \right)$$
$$\approx - \sum_a \sum_{\boldsymbol{x}_{\partial a}} b_a(\boldsymbol{x}_{\partial a}) \log f_a(\boldsymbol{x}_{\partial a}) =: U_{\text{Bethe}}(\{b_a\})$$

$$b(\boldsymbol{x}) \approx \frac{\prod_a b_a(\boldsymbol{x}_{\partial a})}{\prod_i b_i(\boldsymbol{x})^{d_i - 1}}$$

$$H(b) = -\sum_{\boldsymbol{x}} b(x) \log b(x)$$
$$\approx - \sum_{\boldsymbol{x}} b(x) \log \frac{\prod_a b_a(\boldsymbol{x}_{\partial a})}{\prod_i b_i(\boldsymbol{x})^{d_i - 1}}$$
$$= -\sum_a \sum_{\boldsymbol{x}_{\partial a}} b_a(\boldsymbol{x}_{\partial a}) \log b_a(\boldsymbol{x}_{\partial a}) + \sum_i (d_i - 1) \sum_i b_i(x_i) \log b_i(x_i)$$
$$=: H_{\text{Bethe}}(\{b_i\}, \{b_a\})$$

Minimization of Bethe free energy

$$F_{\text{Bethe}}(\{b_i\},\{b_a\}) := U_{\text{Bethe}}(\{b_a\}) - H_{\text{Bethe}}(\{b_i\},\{b_a\})$$

$$\begin{aligned}
\text{minimize}: \quad & F_{\text{Bethe}}(\{b_i\},\{b_a\}) \\
\text{subject to}: \quad & b_i(x_i) \geq 0, \quad \forall i \\
& b_a(\boldsymbol{x}_{\partial a}) \geq 0, \quad \forall a \\
& \sum_i b_i(x_i) = 1 \\
& \sum_a b_a(\boldsymbol{x}_{\partial a}) = 1 \\
& \sum_{\boldsymbol{x}_{\partial a} \backslash x_i} b_a(\boldsymbol{x}_{\partial a}) = b_i(x_i), \quad \forall a, \forall i \in \partial a
\end{aligned}$$

Stationary point of Lagrangian of Bethe free energy

[Yedidia, Freeman, and Weiss 2005]

$$L := F_{\text{Bethe}}(\{b_i\}, \{b_a\}) + \sum_a \gamma_a \left[\sum_{\boldsymbol{x}_{\partial a}} b_a(\boldsymbol{x}_{\partial a}) - 1 \right] + \sum_i \gamma_i \left[\sum_x b_i(x) - 1 \right]$$
$$+ \sum_a \sum_{i \in \partial a} \sum_{x_i} \lambda_{ai}(x_i) \left[b_i(x_i) - \sum_{\boldsymbol{x}_{\partial a} \setminus x_i} b_a(\boldsymbol{x}_{\partial a}) \right]$$

Stationary points of Lagrangian is fixed points of BP

$$b_a(\boldsymbol{x}_{\partial a}) \propto f_a(\boldsymbol{x}_{\partial a}) \prod_{i \in \partial a} m_{i \to a}(x_i)$$
$$b_i(x_i) \propto \prod_{i \in \partial a} m_{a \to i}(x_i)$$

where

$$m_{i \to a}(x_i) \propto \prod_{c \in \partial i \setminus a} m_{c \to i}(x_i)$$
$$m_{a \to i}(x_i) \propto \sum_{\boldsymbol{x}_{\partial a} \setminus x_i} f_a(\boldsymbol{x}_{\partial a}) \prod_{j \in \partial a \setminus i} m_{j \to a}(x_j)$$

Relation between annealed free energy and belief propagation

Random regular factor graph ensemble

Factor graph: bipartite graph which defines probability measure

$$\mu(\boldsymbol{x}) = \frac{1}{Z} \prod_{a} f_a(\boldsymbol{x}_{\partial a})$$

$$Z := \sum_{\boldsymbol{x} \in \mathcal{X}^n} \prod_{a} f_a(\boldsymbol{x}_{\partial a}), \qquad \text{(partition function)}$$

Random (l, r)-regular factor graph ensemble:

l: degree of variable nodes, r: degree of factor nodes

Random ensemble of factor graphs

Annealed free energy

Factor graph: bipartite graph which defines probability measure

$$\mu(\boldsymbol{x}) = \frac{1}{Z} \prod_a f_a(\boldsymbol{x}_{\partial a})$$

$$Z := \sum_{\boldsymbol{x} \in \mathcal{X}^n} \prod_a f_a(\boldsymbol{x}_{\partial a}), \qquad \text{(partition function)}$$

Random (l, r)-regular factor graph ensemble:

l: degree of variable nodes, r: degree of factor nodes

Random ensemble of factor graphs

(Quenched) free energy:

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\log Z]$$

Annealed free energy:

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[Z]$$

Contribution to partition function of particular types

$\{v_x\}_{x\in\mathcal{X}}$: the number of variable nodes of value $x \in \mathcal{X}$ is v_x
$\{u_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathcal{X}^r}$: the number of factor nodes of value $\boldsymbol{x} \in \mathcal{X}^r$ is $u_{\boldsymbol{x}}$

$$Z = \sum_{\boldsymbol{x}\in\mathcal{X}^N} \prod_a f(\boldsymbol{x}_{\partial a})$$
$$= \sum_{\{v\},\{u\}} N(\{v\},\{u\}) \prod_{\boldsymbol{x}\in\mathcal{X}^r} f(\boldsymbol{x})^{u_{\boldsymbol{x}}}.$$

$$\mathbb{E}[N(\{v\},\{u\})] = \binom{N}{\{v_x\}_{x\in\mathcal{X}}} \binom{\frac{l}{r}N}{\{u_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathcal{X}^r}} \frac{\prod_{x\in\mathcal{X}}(v_x l)!}{(Nl)!}.$$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\{\nu\},\{\mu\})]$$
$$= \frac{l}{r}\mathcal{H}(\{\mu\}) - (l-1)\mathcal{H}(\{\nu\}) + \frac{l}{r}\sum_{\boldsymbol{x}\in\mathcal{X}^r} \mu(\boldsymbol{x}) \log f(\boldsymbol{x}).$$

Annealed free energy of fixed type and Bethe free energy

$$F_{\text{Bethe}}(\{b_i\}, \{b_a\}) = -\sum_a \sum_{\boldsymbol{x}_{\partial a}} b_a(\boldsymbol{x}_{\partial a}) \log f_a(\boldsymbol{x}_{\partial a}) + \sum_a \sum_{\boldsymbol{x}_{\partial a}} b_a(\boldsymbol{x}_{\partial a}) \log b_a(\boldsymbol{x}_{\partial a}) - \sum_i (d_i - 1) \sum_i b_i(x_i) \log b_i(x_i)$$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\{\nu\}, \{\mu\})] = \frac{l}{r} \sum_{\boldsymbol{x}\in\mathcal{X}^r} \mu(\boldsymbol{x}) \log f(\boldsymbol{x}) + \frac{l}{r}\mathcal{H}(\{\mu\}) - (l-1)\mathcal{H}(\{\nu\}).$$

Maximization of the exponents of contributions

$$
\begin{aligned}
\text{maximize}: \quad & \frac{l}{r}\mathcal{H}(\{\mu\}) - (l-1)\mathcal{H}(\{\nu\}) + \frac{l}{r}\sum_{\boldsymbol{x}\in\mathcal{X}^r}\mu(\boldsymbol{x})\log f(\boldsymbol{x}) \\
\text{subject to}: \quad & \nu(x) \geq 0, \quad \forall x \in \mathcal{X} \\
& \mu(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{X}^r \\
& \sum_{x\in\mathcal{X}} \nu(x) = 1 \\
& \sum_{\boldsymbol{x}\in\mathcal{X}^r} \mu(\boldsymbol{x}) = 1 \\
& \frac{1}{r}\sum_{k=1}^{r}\sum_{\substack{\boldsymbol{x}\setminus x_k \\ x_k = z}} \mu(\boldsymbol{x}) = \nu(z), \quad \forall z \in \mathcal{X}
\end{aligned}
$$

The stationary condition

The stationary condition is

$$\nu(x) \propto m_{f\to v}(x)^{l}$$
$$\mu(\boldsymbol{x}) \propto f(\boldsymbol{x}) \prod_{i=1}^{r} m_{v\to f}(x_i)$$

where

$$m_{v\to f}(x) \propto m_{f\to v}(x)^{l-1}$$
$$m_{f\to v}(x) \propto \sum_{k=1}^{r} \sum_{\substack{\boldsymbol{x}\setminus x_k \\ x_k = x}} f(\boldsymbol{x}) \prod_{j\neq k} m_{v\to f}(x_j).$$

If $f(\boldsymbol{x})$ is invariant under any permutation of $\boldsymbol{x} \in \mathcal{X}^r$

$$m_{f\to v}(x) \propto \sum_{\substack{\boldsymbol{x}\setminus x_1 \\ x_1 = x}} f(\boldsymbol{x}) \prod_{j\neq 1} m_{v\to f}(x_j).$$

Annealed free energy

Theorem 1.

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z] = \max_{(m_{f\to v}, m_{v\to f})\in\mathcal{S}} \left\{ \frac{l}{r} \log Z_f + \log Z_v - l \log Z_{fv} \right\}.$$

where $\mathcal{S}$ denotes the set of saddle points, and where

$$Z_v := \sum_x m_{f\to v}(x)^l$$

$$Z_f := \sum_{\boldsymbol{x}} f(\boldsymbol{x}) \prod_{i=1}^{r} m_{v\to f}(x_i)$$

$$Z_{fv} := \sum_x m_{f\to v}(x) m_{v\to f}(x).$$

Number of solutions

If

$$\sum_{k=1}^{r} \sum_{\substack{\boldsymbol{x} \setminus x_k \\ x_k = x}} f(\boldsymbol{x})$$

is constant among all $x \in \mathcal{X}$, the uniform message $m_{f \to v}(x)$、$m_{v \to f}(x)$ is a saddle point.
The contribution of the uniform message is

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[Z(\nu, \mu)] = \log q + \frac{l}{r} \log \left(\frac{N_f}{q^r} \right) \qquad \text{(design rate)}$$

where $q := |\mathcal{X}|$、$N_f := \sum_{\boldsymbol{x}} f(\boldsymbol{x})$.
For CSP i.e., $f(\boldsymbol{x}) \in \{0, 1\}$, the expected number of solutions is about

$$q^N \left(\frac{N_f}{q^r} \right)^{\frac{l}{r} N}.$$

This intuitively means all constraints are independent.

Contribution to partition function of fixed variable type

$$Z(\{\nu\}) := \sum_{\{\mu\}} Z(\{\nu\}, \{\mu\})$$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\{\nu\})]$$

$$= \sup_{\{\mu\}} \left\{ \frac{l}{r}\mathcal{H}(\{\mu\}) - (l-1)\mathcal{H}(\{\nu\}) + \frac{l}{r} \sum_{\boldsymbol{x}\in\mathcal{X}^r} \mu(\boldsymbol{x}) \log f(\boldsymbol{x}) \right\}$$

where $\{\mu\}$ satisfies

$$\mu(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{X}^r$$

$$\sum_{\boldsymbol{x}\in\mathcal{X}^r} \mu(\boldsymbol{x}) = 1$$

$$\frac{1}{r} \sum_{k=1}^{r} \sum_{\substack{\boldsymbol{x}\setminus x_k \\ x_k = z}} \mu(\boldsymbol{x}) = \nu(z), \quad \forall z \in \mathcal{X}$$

Convex optimization problem with linear constraints.

The stationary condition

The stationary condition is

$$\mu(\boldsymbol{x}) \propto f(\boldsymbol{x}) \prod_{i=1}^{r} m_{v \to f}(x_i)$$

where

$$\begin{aligned} \nu(x) &\propto h(x) m_{f \to v}(x)^{l} \\ m_{v \to f}(x) &\propto h(x) m_{f \to v}(x)^{l-1} \\ m_{f \to v}(x) &\propto \sum_{k=1}^{r} \sum_{\substack{\boldsymbol{x} \setminus x_k \\ x_k = x}} f(\boldsymbol{x}) \prod_{j \neq k} m_{v \to f}(x_j). \end{aligned}$$

If $f(\boldsymbol{x})$ is invariant under any permutation of $\boldsymbol{x} \in \mathcal{X}^r$

$$m_{f \to v}(x) \propto \sum_{\substack{\boldsymbol{x} \setminus x_1 \\ x_1 = x}} f(\boldsymbol{x}) \prod_{j \neq 1} m_{v \to f}(x_j).$$

Growth rate of contribution to partition function of fixed variable type

Theorem 2.

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\{\nu\})]$$
$$= \max_{(m_{f\to v}, m_{v\to f})\in\mathcal{S}} \left\{ \frac{l}{r} \log Z_f + \log Z_v - l \log Z_{fv} - \sum_x \nu(x) \log h(x) \right\}$$

where $\mathcal{S}$ denotes the set of saddle points, and where

$$Z_v := \sum_x h(x) m_{f\to v}(x)^l$$
$$Z_f := \sum_{\boldsymbol{x}} f(\boldsymbol{x}) \prod_{i=1}^{r} m_{v\to f}(x_i)$$
$$Z_{fv} := \sum_x m_{f\to v}(x) m_{v\to f}(x).$$

Growth rate of regular LDPC codes

$$G(\omega) = \frac{l}{r}\log\frac{1+z^r}{2} + \log\left[e^h\left(\frac{1+y}{2}\right)^l + e^{-h}\left(\frac{1-y}{2}\right)^l\right] - l\log\frac{1+yz}{2} - \omega' h$$

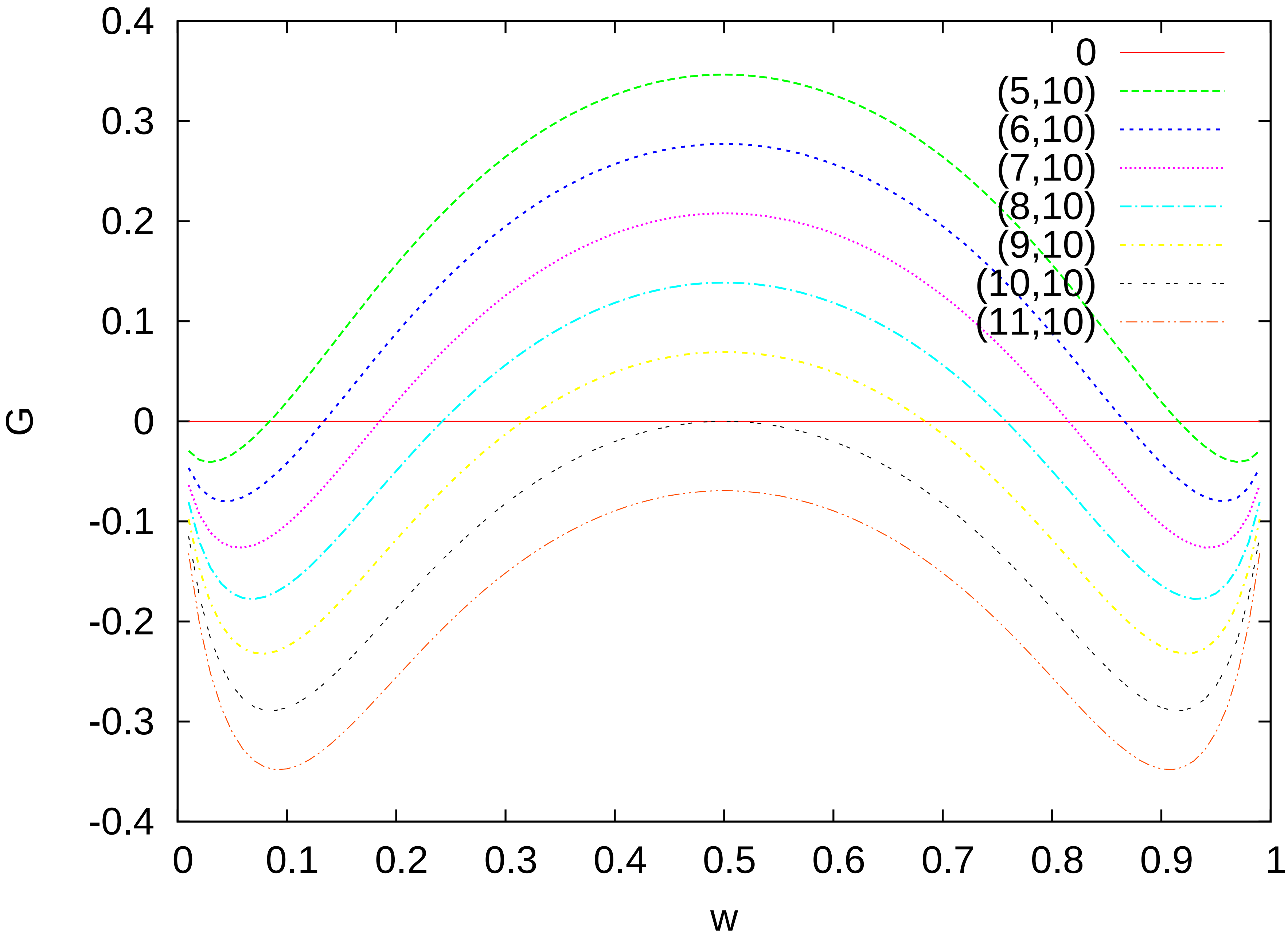
where $\omega' := 1-2\omega$ and

$$\omega' = \tanh(h + l\tanh^{-1}(y))$$
$$y = z^{r-1}$$
$$z = \tanh(h + (l-1)\tanh^{-1}(y)).$$

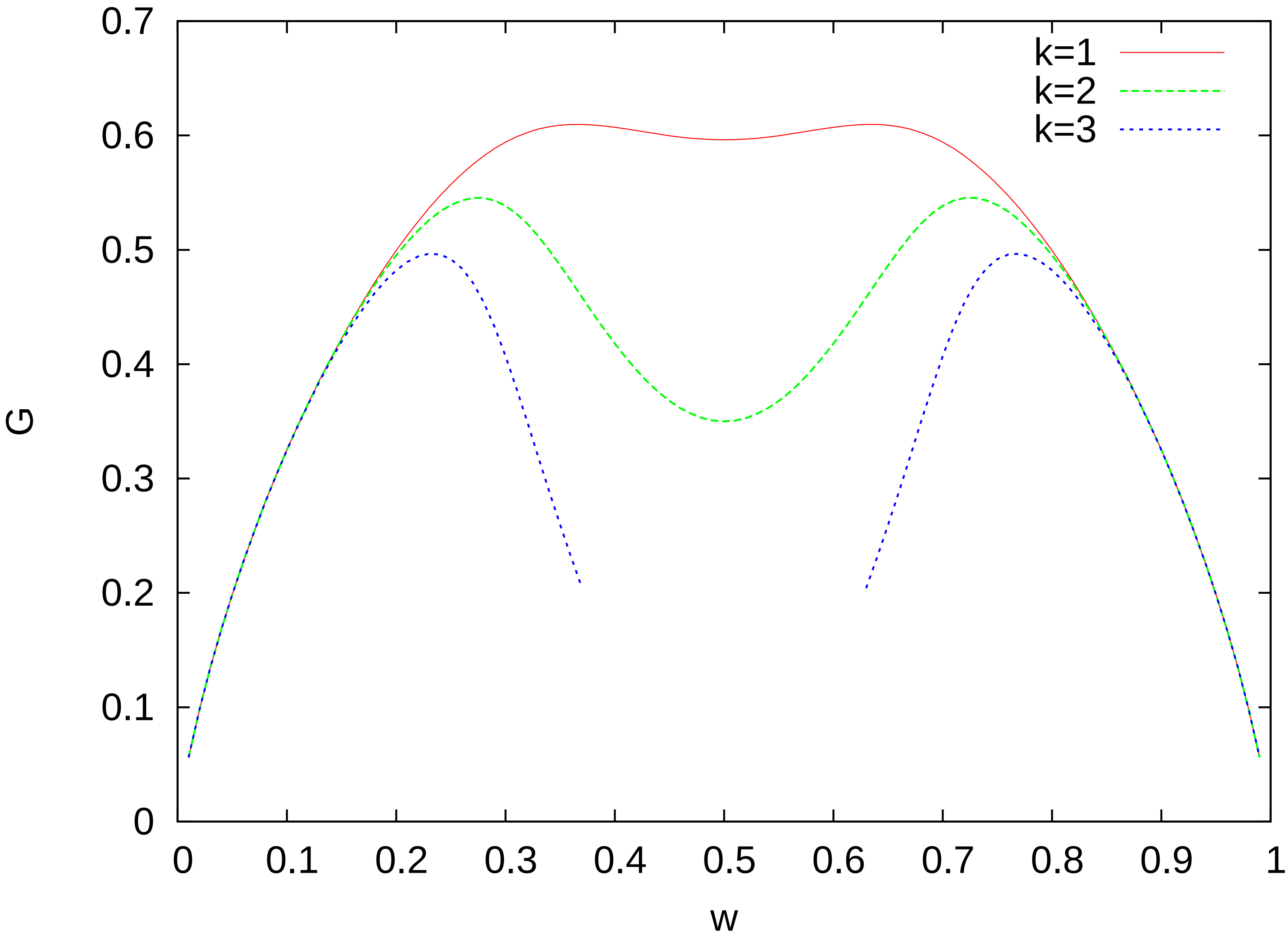
This result can be easily understood from correspondings

$$\omega' = \nu(0) - \nu(1)$$
$$y = m_{f\to v}(0) - m_{f\to v}(1)$$
$$z = m_{v\to f}(0) - m_{v\to f}(1)$$
$$h(x) = e^{(-1)^x h}$$

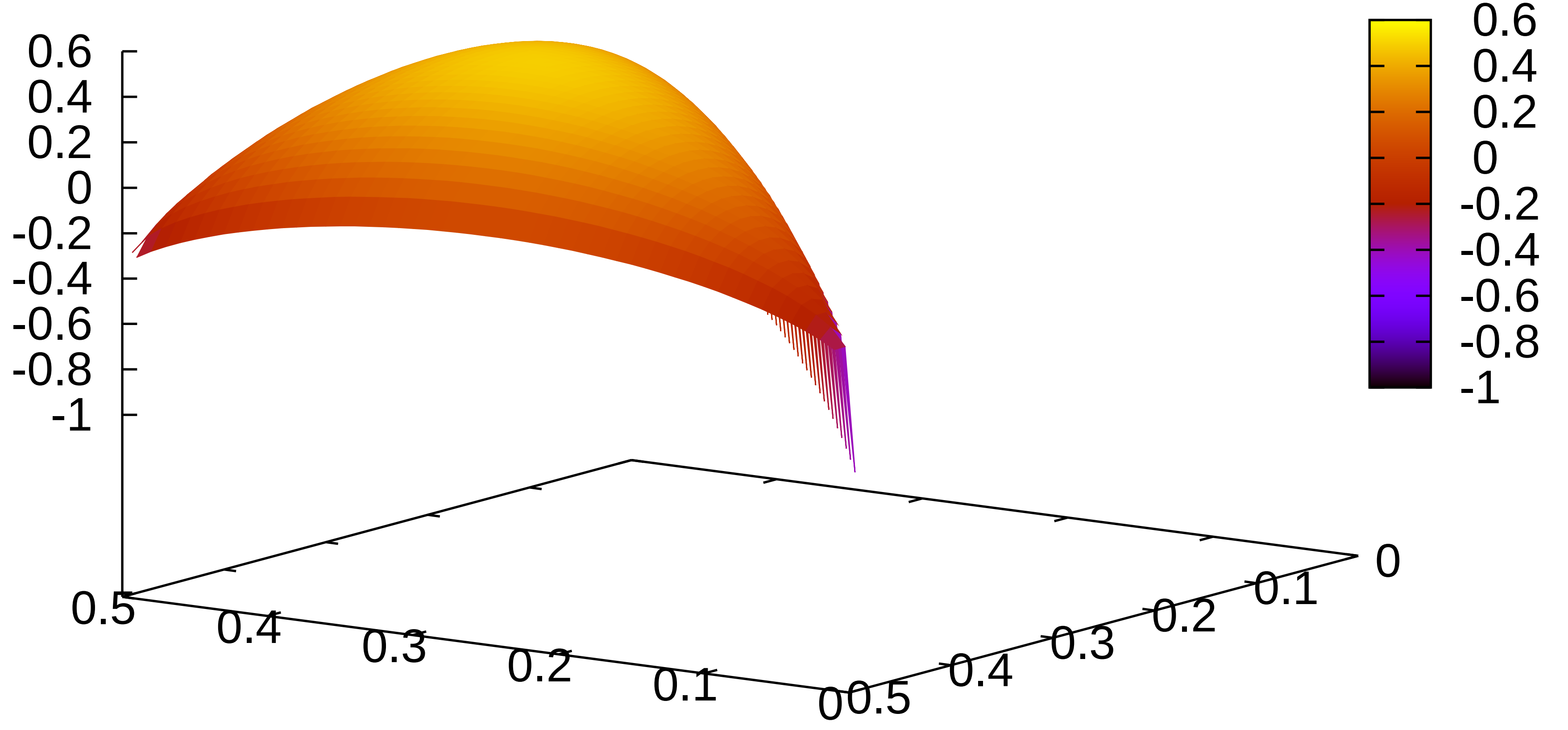
Growth rate of regular LDPC codes



Growth rate of binary CSP



Growth rate of (3,2)-regular-3-coloring



Replica theory

This story continues into the replica theory
(see the paper in arXiv).
But, we don't deal with it here.

$$\mathbb{E}[\log Z] = \left.\frac{\partial \log \mathbb{E}[Z^n]}{\partial n}\right|_{n=0}$$

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\log Z] &= \lim_{N\to\infty} \frac{1}{N} \lim_{n\to 0} \frac{\log \mathbb{E}[Z^n]}{n} \\ &\stackrel{?}{=} \lim_{n\to 0} \frac{1}{n} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z^n]\end{aligned}$$

The replica method is methematically not rigorous e.g., exchange of limits, analytic continuation of n.

Growth rate of spatially coupled LDPC codes and threshold saturation phenomenon

Protograph ensemble

The similar results also holds for protograph ensemble [Vontobel 2010]

In this morning, Kenta has explained

- Definition of protograph ensemble
- Definition of spatially coupled LDPC codes
- Threshold saturation phenomenon of EXIT curve

Growth rate of spatially coupled LDPC codes

$$G(\omega) = \frac{1}{2L+1}\left[\frac{l}{r}\sum_{j=-L}^{L+l-1}\log\left(\log\frac{1+\prod_{k=0}^{l-1} z_{j,k}{}^{\frac{r}{l}}}{2}\right)\right.$$

$$+\sum_{i=-L}^{L}\log\left[e^{h}\prod_{k=0}^{l-1}\left(\frac{1+y_{i,k}}{2}\right)+e^{-h}\prod_{k=0}^{l-1}\left(\frac{1-y_{i,k}}{2}\right)\right]$$

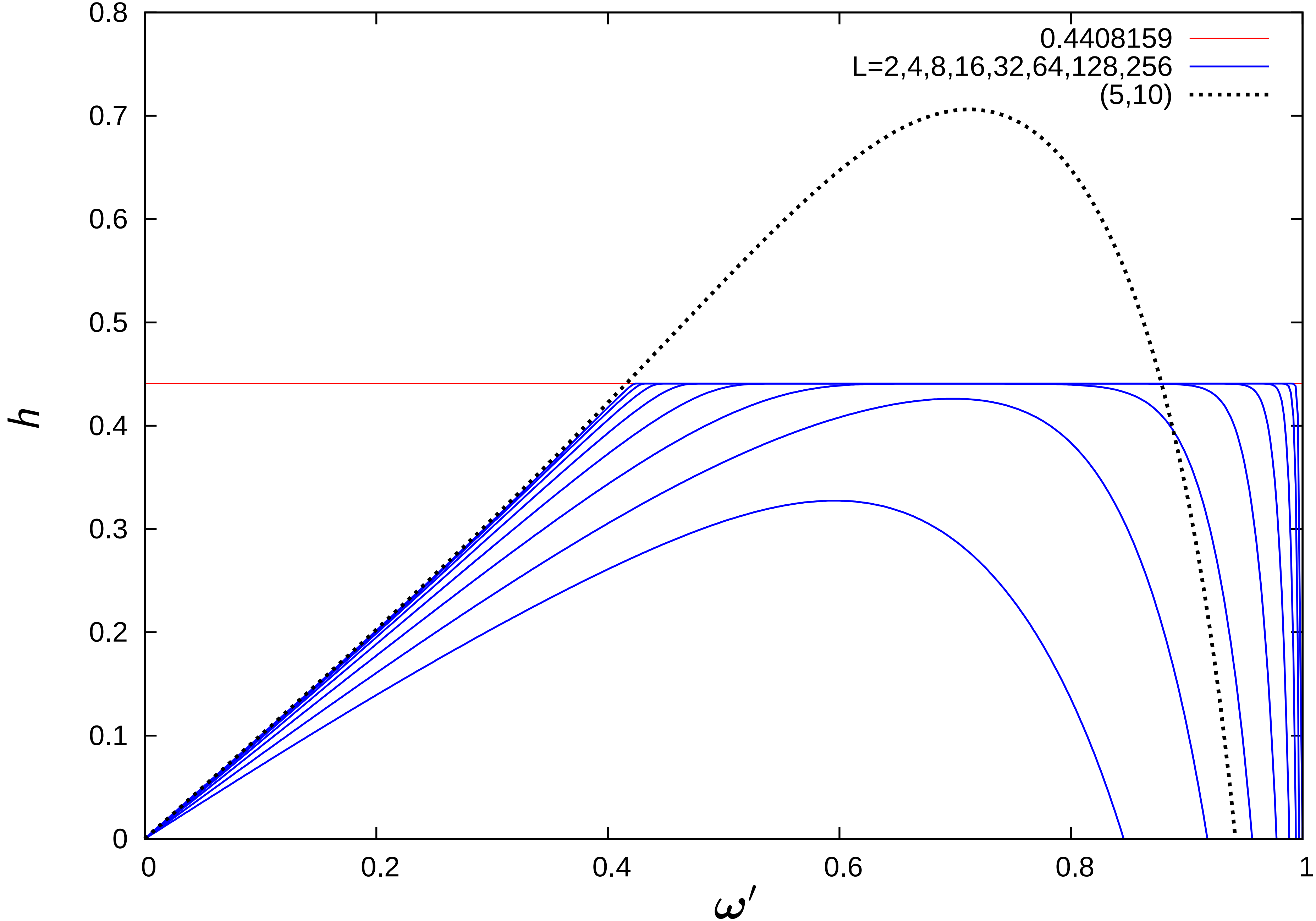
$$\left.-\sum_{i=-L}^{L}\sum_{k=0}^{l-1}\log\left(\frac{1+y_{i,k}z_{i+k,k}}{2}\right)\right]-\omega' h.$$

$$\omega' = \frac{1}{2L+1}\sum_{i=-L}^{L}\tanh\left(h+\sum_{k=0}^{l-1}\tanh^{-1}\left(y_{i,k}\right)\right)$$

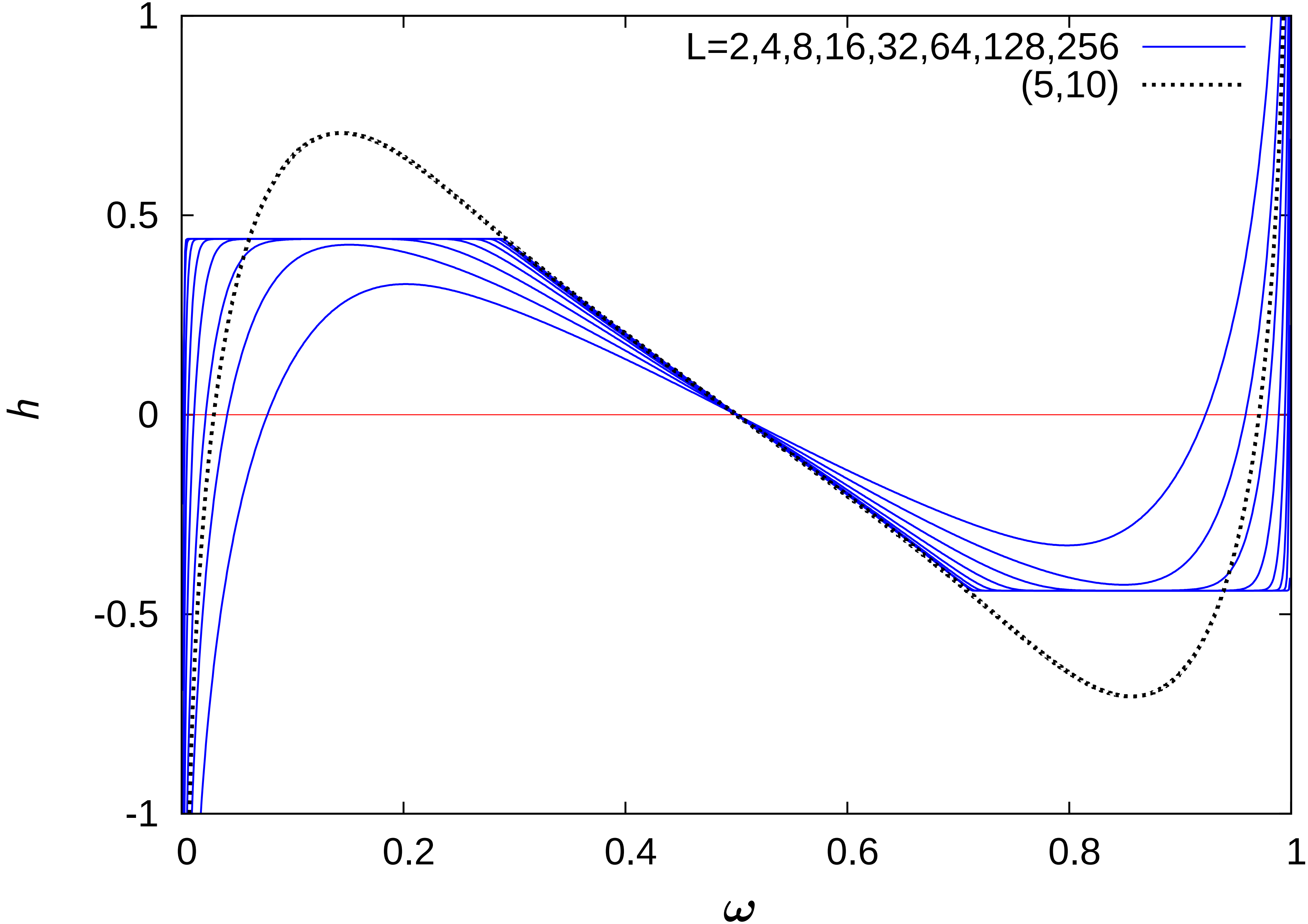
$$z_{j,k} = \tanh\left(h+\sum_{k'=0,k'\neq k}^{l-1}\tanh^{-1}\left(y_{j-k,k'}\right)\right)$$

$$y_{i,k} = z_{i+k,k}{}^{\frac{r}{l}-1}\prod_{k'=0,k'\neq k}^{l-1} z_{i+k,k'}{}^{\frac{r}{l}}$$

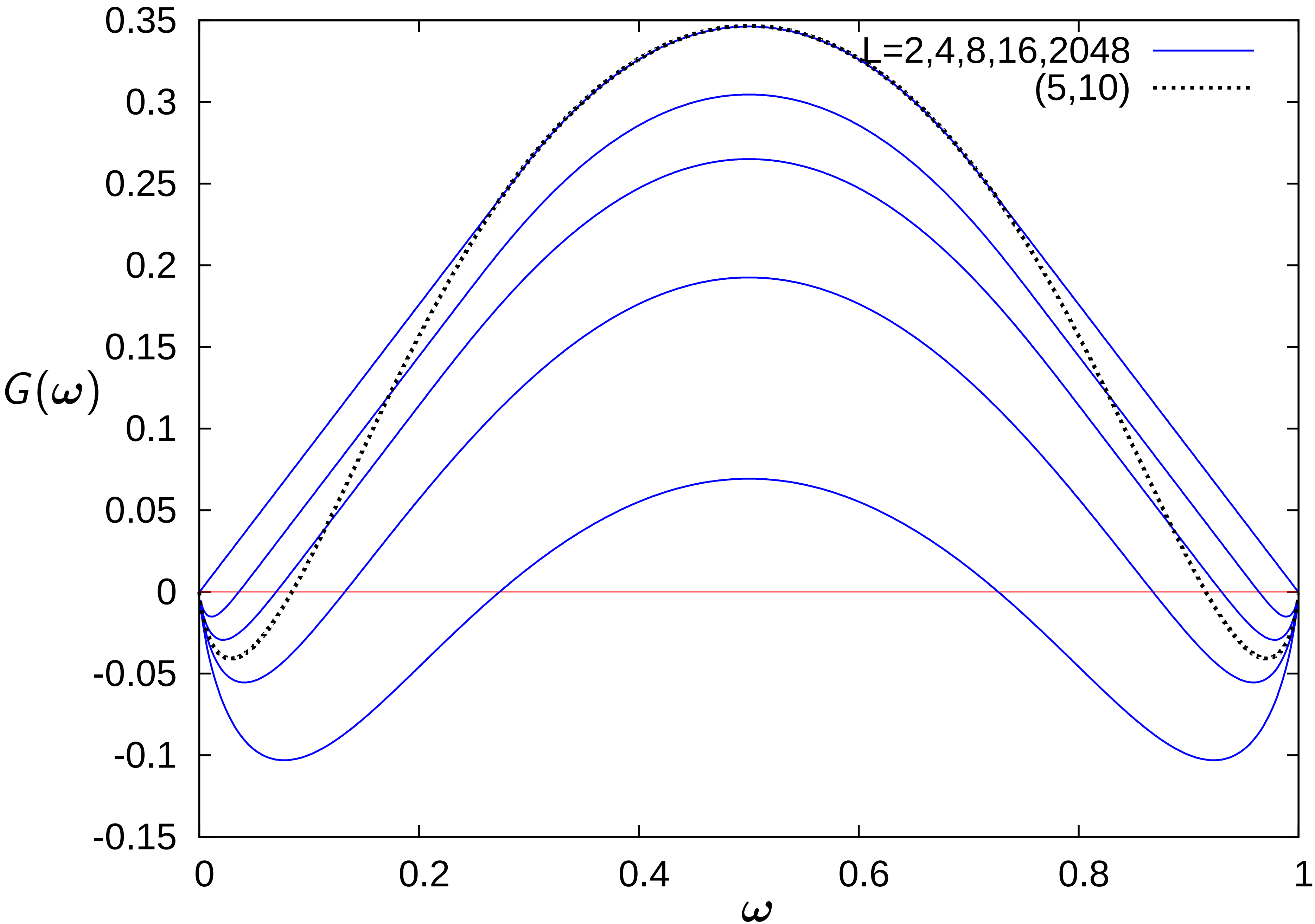
ω' versus h



ω versus h: Derivative of growth rate



Growth rate



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
$G(\omega)$
0
0.2
0.4
0.6
0.8
1
ω
L=2,4,8,16,2048
(5,10)

Conclusion

- Contribution to annealed free energy of particular type has similar form of Bethe free energy.
- The stationary condition of maximization problem for annealed free energy is similar to equation of belief propagation.
- There exists threshold saturation phenomenon in the calculation of growth rate of spatially coupled LDPC codes.
- We now can calculate annealed free energy of any coupled factor graphs. Effect of boundary condition is not obvious. BP iterations does not necessarily converge (even for uncoupled cases).

Acknowledgment

The basic idea that the growth rate approaches to the concave hull is given by Nicolas Macris. I acknowledges Hamed Hassani and Toshiyuki Tanaka for encouragement and discussion.

- arXiv:1102.3132, paper about connection between Bethe and annealed free energies (submitted to ISIT 2011).
- Joint paper with Hamed and Nicolas about growth rate of coupled LDPC codes was submitted to ISIT 2011.