Finite-Length Analysis of Irregular Expurgated LDPC Codes under Finite Number of Iterations

Ryuhei Mori   Toshiyuki Tanaka   Kenta Kasai
Kohichi Sakaniwa

ISIT2009

LDPC codes over the binary erasure channel

The aim of our research

To estimate the bit error probability $P_b(n, \epsilon, t)$ of LDPC codes

over the BEC under belief propagation decoding

where

- n: blocklength
- ϵ: erasure probability of BEC
- t: the number of iterations

Previous Results

Analysis for the BEC

Exact or Asymptotic	Blocklength	Number of Iterations	Computational Complexity	Irregular Ensembles	Method
Exact	∞	t	$O(t)$	○	Density Evolution [1]
Exact	n	∞	$O(n^3)$	△	Stopping Sets [2]
Asymptotic	n	∞	$O(1)$	○	Scaling Law [3]
Asymptotic	n	t	$O(t^3)$	× $\rightarrow$ ○	This Research

[1] Richardson and Urbanke 2001
[2] Di et al. 2002
[3] Amraoui et al. 2004

The Main Result of This Work

Our result presented in ISIT2008 is generalized for irregular ensembles

Asymptotic Expansion

Asymptotic Expansion w.r.t n while t is fixed

$$\mathrm{P_b}(n, \epsilon, t) = \mathrm{P_b}(\infty, \epsilon, t) + \alpha(\epsilon, t)\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Coefficient of $1/n$

$$\alpha(\epsilon, t) := \lim_{n \longrightarrow \infty} n\left(\mathrm{P_b}(n, \epsilon, t) - \mathrm{P_b}(\infty, \epsilon, t)\right)$$

Approximation

$$\mathrm{P_b}(n, \epsilon, t) \approx \mathrm{P_b}(\infty, \epsilon, t) + \alpha(\epsilon, t)\frac{1}{n}$$

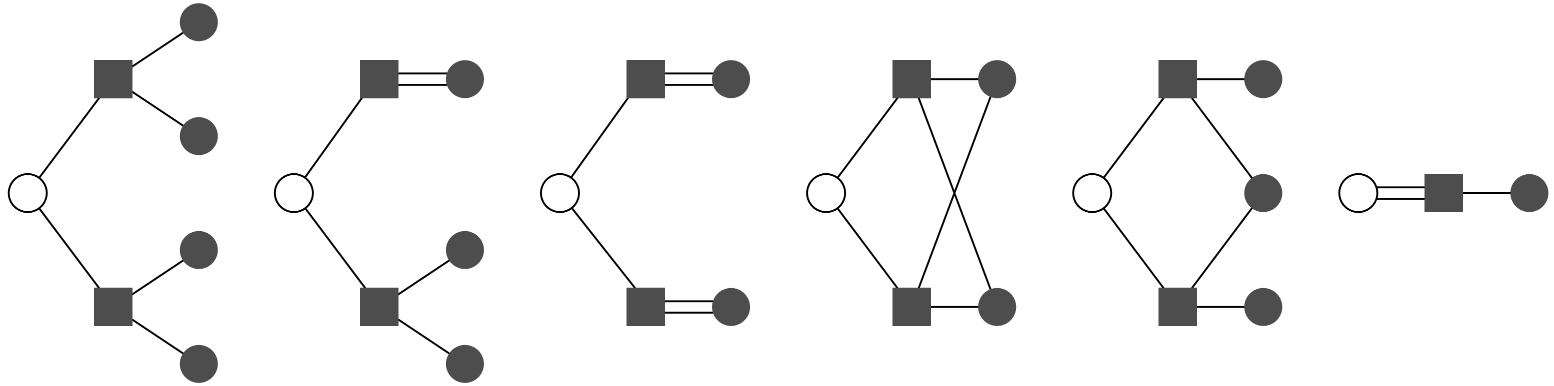
Our purpose is to derive $\alpha(\epsilon, t)$ for irregular ensembles

Neighborhoods

$$\mathrm{P_b}(n,\epsilon,t) = \sum_{G \,\in\, \text{the set of all neighborhoods of depth } t} \mathbb{P}_n(G)\,\mathrm{P_b}(G,\epsilon)$$

$\mathrm{P_b}(G,\epsilon)$	$\varepsilon(1-(1-\varepsilon)^2)^2$	$\varepsilon^2(1-(1-\varepsilon)^2)$	ε^3	$\varepsilon(1-(1-\varepsilon)^2)$	$\varepsilon^2(1+\varepsilon(1-\varepsilon))$	ε
$\mathbb{P}_n(G)$	$\frac{(2n-6)(2n-8)}{(2n-1)(2n-5)}$	$\frac{2(2n-6)}{(2n-1)(2n-5)}$	$\frac{1}{(2n-1)(2n-5)}$	$\frac{2}{(2n-1)(2n-5)}$	$\frac{4(2n-6)}{(2n-1)(2n-5)}$	$\frac{2}{(2n-1)}$
Order of $\mathbb{P}_n(G)$	1	n^{-1}	n^{-2}	n^{-2}	n^{-1}	n^{-1}

G



Number of cycles

The basic fact

If G has k cycles

$$\mathbb{P}_n(G) = \Theta(n^{-k}).$$

The large blocklength limit of the bit error probability

$$\mathrm{P_b}(\infty, \epsilon, t) = \lim_{n \to \infty} \sum_{G \in \text{ the set of all neighborhoods of depth } t} \mathbb{P}_n(G)\mathrm{P_b}(G, \epsilon)$$

$$= \lim_{n \to \infty} \sum_{G \in \text{ the set of all cycle-free neighborhoods of depth } t} \mathbb{P}_n(G)\mathrm{P_b}(G, \epsilon) \quad .$$

Calculation of $\alpha(\epsilon, t)$

$$\alpha(\epsilon, t) := \lim_{n\to\infty} n(\mathrm{P_b}(n, \epsilon, t) - \mathrm{P_b}(\infty, \epsilon, t))$$

$$= \underbrace{\lim_{n\to\infty} n \left(\sum_{G \in \text{the set of all cycle-free neighborhoods of depth } t} \mathbb{P}_n(G)\mathrm{P_b}(G, \epsilon) - \mathrm{P_b}(\infty, \epsilon, t) \right)}_{\beta(\epsilon, t)}$$

$$+ \underbrace{\lim_{n\to\infty} n \sum_{G \in \text{the set of all single-cycle neighborhoods of depth } t} \mathbb{P}_n(G)\mathrm{P_b}(G, \epsilon)}_{\gamma(\epsilon, t)} .$$

In the previous work [Mori et al., ISIT2008],
$\gamma(\epsilon, t)$ was obtained for irregular ensembles
but $\beta(\epsilon, t)$ was obtained only for regular ensembles

Contribution of Cycle-Free Neighborhoods

$$\beta(\epsilon, t) =$$

$$\frac{1}{2L'(1)}\left(\mathbb{E}_t[K(K-1)P] - \sum_i \frac{i}{\lambda_i}\mathbb{E}_t[V_i(V_i-1)P] - \sum_j \frac{j}{\rho_j}\mathbb{E}_t[C_j(C_j-1)P]\right)$$

The expectations are taken on the tree ensemble of depth t

$$\mathbb{P}_\infty(G) := \lim_{n\longrightarrow\infty} \mathbb{P}_n(G)$$

- K: the number of edges in a tree neighborhood
- V_i: the number of variable nodes of degree i in a tree neighborhood
- C_j: the number of check nodes of degree j in a tree neighborhood
- P: the erasure probability of the root node after BP decoding on a tree neighborhood

Method of Generating Function

$$\mathbb{E}_t[K(K-1)P] = \left.\frac{\partial^2 \mathbb{E}_t[x^K P]}{\partial x^2}\right|_{x=1}$$

$$\mathbb{E}_t[V_i(V_i-1)P] = \left.\frac{\partial^2 \mathbb{E}_t[x^{V_i} P]}{\partial x^2}\right|_{x=1}$$

$$\mathbb{E}_t[C_j(C_j-1)P] = \left.\frac{\partial^2 \mathbb{E}_t[x^{C_j} P]}{\partial x^2}\right|_{x=1}$$

$$\mathbb{E}_t[x^K P] = \frac{1}{x}\,\mathbb{E}_t\left[\prod_k y_k^{V_k} \prod_l z_l^{C_l} P\right]\Bigg|_{y_k = x,\, z_l = x \text{ for all } k,\, l}$$

$$\mathbb{E}_t[x^{V_i} P] = \mathbb{E}_t\left[\prod_k y_k^{V_k} \prod_l z_l^{C_l} P\right]\Bigg|_{y_i = x,\, y_k = 1,\, z_l = 1 \text{ for all } k \neq i,\, l}$$

$$\mathbb{E}_t[x^{C_j} P] = \mathbb{E}_t\left[\prod_k y_k^{V_k} \prod_l z_l^{C_l} P\right]\Bigg|_{z_j = x,\, y_k = 1,\, z_l = 1 \text{ for all } k,\, l \neq j}$$

The Mother Generating Function

$$\mathbb{E}_t\left[\prod_k y_k^{V_k}\prod_l z_l^{C_l}P\right] = \epsilon\mathfrak{L}(F(t)),$$

where

$$F(t) := \begin{cases} 1, & \text{if } t = 0 \\ \mathcal{P}(g(t)) - \mathcal{P}(G(t)), & \text{otherwise,} \end{cases}$$

$$G(t) := \mathcal{L}(f(t-1)) - \epsilon\mathcal{L}(F(t-1)),$$

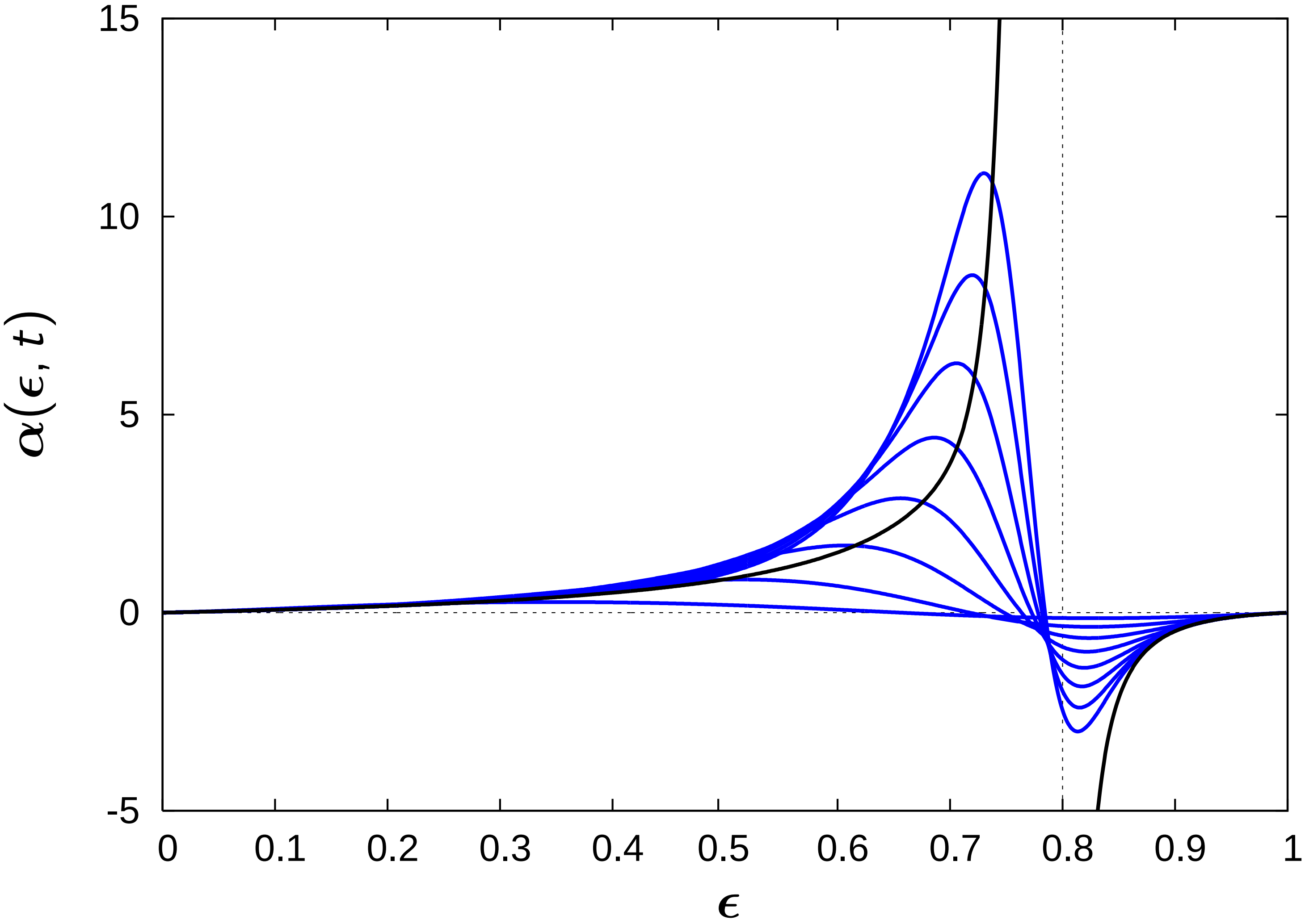
$$f(t) := \begin{cases} 1, & \text{if } t = 0 \\ \mathcal{P}(g(t)), & \text{otherwise,} \end{cases}$$

$$g(t) := \mathcal{L}(f(t-1)),$$

and where

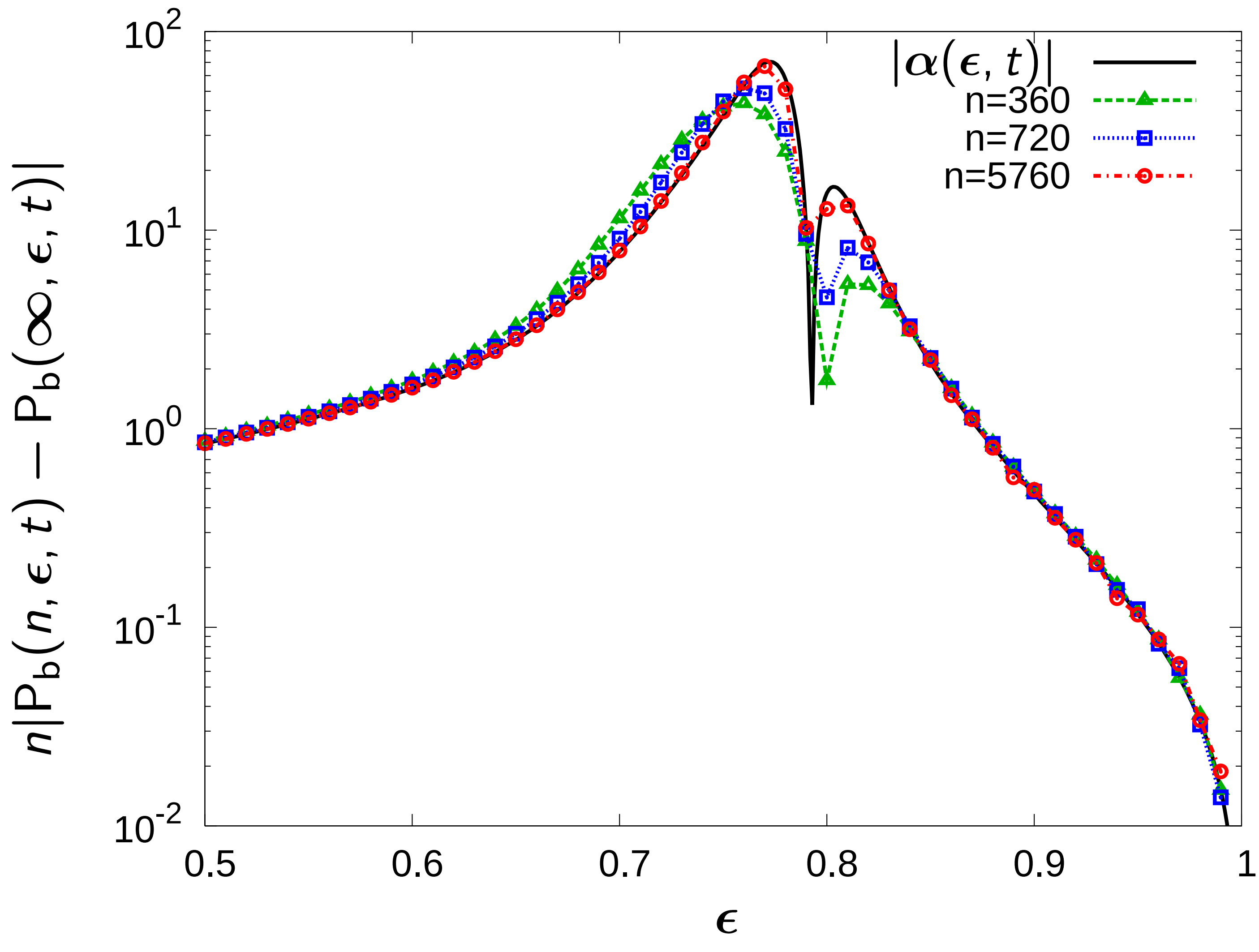
$$\mathfrak{L}(x) := \sum_i L_i y_i x^i, \qquad \mathcal{L}(x) := \sum_i \lambda_i y_i x^{i-1}, \qquad \mathcal{P}(x) := \sum_j \rho_j z_j x^{j-1}.$$

$\alpha(\epsilon, t)$ for Optimized Irregular Ensemble



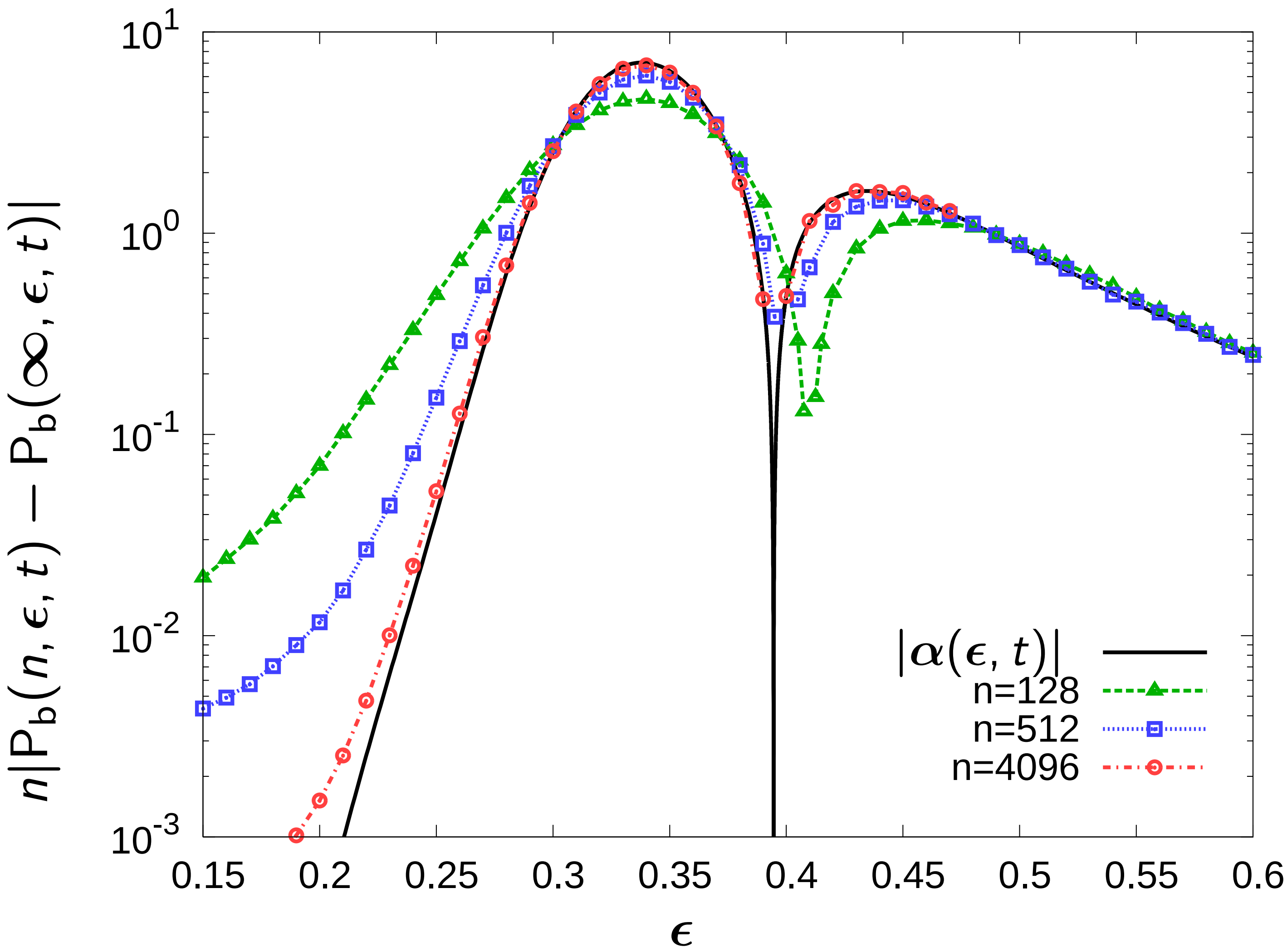
$\lambda(x) = 0.500x + 0.153x^2 + 0.112x^3 + 0.055x^4 + 0.180x^8$
$\rho(x) = 0.492x^2 + 0.508x^3$ $R \approx 0.192$, $\epsilon_{\mathrm{BP}} \approx 0.8$, $t = 1, 2, \ldots, 8,\ 50$

Simulation Results



$\lambda(x) = 0.500x + 0.153x^2 + 0.112x^3 + 0.055x^4 + 0.180x^8$
$\rho(x) = 0.492x^2 + 0.508x^3$ $R \approx 0.192,\ \epsilon_{BP} \approx 0.8,\ t = 20$

Ensembles with $\lambda_2 = 0$



(3, 6)-regular ensemble $\quad t = 5 \quad \mathrm{P_b}(n,\epsilon,\infty) = \Theta(1/n^2)$ for $\epsilon < \epsilon_{\mathrm{BP}}$

For small ϵ, the small number of iteration is sufficient

unless blocklength is sufficiently large

The Speed of Convergence

For the irregular ensemble,
$\lambda(x) = 0.500x + 0.153x^2 + 0.112x^3 + 0.055x^4 + 0.180x^8$
$\rho(x) = 0.492x^2 + 0.508x^3$
when $t = 20$, $n = 5760$,

$$\alpha(\epsilon, t) \approx n\left(\mathrm{P_b}(n, \epsilon, t) - \mathrm{P_b}(\infty, \epsilon, t)\right)$$

for any ϵ (Generally, λ_2 is larger and larger, the convergence is faster)

$\alpha(\epsilon, t)$ consists of
contributions of cycle-free neighborhoods and single-cycle neighborhoods

But the number of variable nodes in the smallest tree of depth 20 is $4194302 \gg 5760$

The probability of cycle-free and single-cycle neighborhoods is zero

Open problem: Why is the speed of the convergence fast?

Conclusion and Open Problems

Conclusion

- Using the generating function method,
  $\beta(\epsilon, t)$ is obtained for irregular ensembles
- The speed of the convergence to $\alpha(\epsilon, t)$ is fast

Open problems

- The fast convergence to $\alpha(\epsilon, t)$ except for
  ensembles with $\lambda_2 = 0$ and ϵ is small
- Minimization of $\mathrm{P_b}(n, \epsilon, t) + \alpha(\epsilon, t)/n$ on some conditions
- Higher order terms i.e. coefficient of $1/n^2$, $1/n^3$ ...
- The limit parameter $\alpha(\epsilon, \infty)$ for irregular ensembles
- Generalization to arbitrary binary memoryless symmetric channels
- Asymptotic analysis of performance based on other limits e.g. $n \longrightarrow \infty$ and $t \longrightarrow \infty$ simultaneously